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Conjugate natural convection heat transfer between two porous media separated by a horizontal wall

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Abstract—The conjugate heat transfer across a thin horizontal wall separating two fluid-saturated porous media at different temperatures is investigated numerically and asymptotically. The solution for large Rayleigh numbers is shown to depend on two nondimensional parameters: α, the ratio of the thermal resistance of the boundary layer in the hot medium to the thermal resistance of the wall, and β, the ratio of the thermal resistances of the boundary layers in the two media. The overall Nusselt number is an increasing function of α tending to zero for $\alpha \to 0$, and to a finite maximum value for $\alpha \to \infty$. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

Natural convection heat transfer in fluid-saturated porous media is of interest to a host of applications, ranging from geophysical problems to thermal insulation, chemical reactors or underground spread of pollutants (see Gebhart *et al.* [1] and Nield and Bejan [2] for recent reviews). Conjugate effects, whereby convection in the porous media is coupled to conduction in the finite-thickness solids bounding these media, are involved in many of these applications, and their realistic modelling often poses problems somewhat more complicated than the classic idealizations of a solid with predetermined temperature or heat flux. A theoretical analysis of the basic heat transfer mechanisms for a vertical conducting plate separating two porous media kept at different temperatures was presented recently [3], following a similar analysis by Treviño *et al.* [4] for Newtonian fluids. In these works, the effects of longitudinal and transverse conduction in the solid were studied by means of asymptotic and numerical methods for large Rayleigh number flows, the solutions were classified in terms of three main non-dimensional parameters, and the dependence of the overall rate of heat transfer on the thermal resistances of the solid and the boundary layers in the fluids was determined. In this paper the same techniques are applied to the case of a horizontal wall separating two porous media. The physical mechanisms retained in the analysis, which corresponds to the asymptotic limit of large Rayleigh numbers, are heat conduction in the solid, and conduction and convection in the porous media. Darcy's law and the Boussinesq approximation are used.

2. FORMULATION AND ASYMPTOTIC ANALYSIS

Consider a horizontal heat-conducting strip of width $2L$ and thickness h in an otherwise adiabatic wall separating two fluid-saturated porous media at temperatures $T_{1\infty}$ and $T_{2\infty} < T_{1\infty}$. Due to the heat conduction across the strip, temperature differences of order $\Delta T = T_{1\infty} - T_{2\infty}$ appear in the porous media that induce natural convection flows. An order of magnitude analysis (e.g. Gebhart *et al.* [1]) shows that, for large values of the Rayleigh number, these motions occur in boundary layers of thickness $L/Ra_i^{1/3}$ on the sides of the strip, where the characteristic velocity is $Ra_i^{2/3}\,\alpha_i/L$. Here $Ra_i = g\beta_i \Delta T K_i L/\alpha_i \nu_i$ are the Rayleigh numbers, with $i = 1$, 2 denoting the hot and cold media, respectively, g is the acceleration of gravity, and K_i, α_i, β_i and ν_i are the permeabilities and thermal diffusivities of the media, and the thermal expansion coefficients and kinematic viscosities of the fluids.

It should be noticed here that the flow in both boundary layers is directed toward the centre of the strip when the hot medium is below the wall, as sketched by the solid arrows in Fig. 1, leading to two vertical plumes not represented in Fig. 1. A standard order of magnitude analysis shows that the thickness and vertical velocity of the plumes at a distance $|y|$ from the wall are of orders $L(y/L)^{2/3}/Ra_i^{4/9}$ and $Ra_i^{8/9}(\alpha_i/L)/(|y|/L)^{1/3}$, respectively, and, therefore, that the entrainment of these plumes induces velocities of order $Ra_i^{4/9}\alpha_i/L$ at distances of the wall of order L. At the asymptotic limit $Ra_i \to \infty$ considered in this work, such velocities are small compared to the velocities in the boundary layers, and therefore the presence of the plumes does not alter the flow in the

NOMENCLATURE

h	thickness of the strip	Greek symbols	
k_i	thermal conductivity of medium i	α	heat conduction parameter, defined in equation (14)
k_w	thermal conductivity of solid	α_i	equivalent thermal diffusivity of medium i
L	half-width of the strip	β	ratio of boundary layer thermal resistances, defined in equation (14)
$\overline{Nu}$	overall Nusselt number defined in equation (15)	ε	plate aspect ratio
$\bar{q}_w$	overall heat flux from medium i to wall	θ_i	non-dimensional temperature of medium i, defined in equation (1)
Ra_i	Rayleigh number of medium i	θ_w	non-dimensional temperature of plate, defined in equation (2)
$T_{i\infty}$	temperature of medium i far from plate	ψ_i^*	stream function for medium i
x^*, y^*	horizontal and vertical Cartesian coordinates.		

boundary layers in first approximation. The correction, however, is of order $Ra_i^{-2/9}$, which may be important for some Rayleigh numbers of practical interest.

Appropriate non-dimensional variables to describe the flows in the boundary layers are

$$x = \frac{x^*}{L} \quad y_1 = Ra_1^{1/3}\frac{y^*-h/2}{L}$$

$$y_2 = -Ra_2^{1/3}\frac{y^*+h/2}{L} \quad \psi_i = \frac{\psi_i^*}{\alpha_i Ra_i^{1/3}}$$

$$\theta_1 = \frac{T_{1\infty}-T_1}{\Delta T} \quad \theta_2 = \frac{T_2 - T_{2\infty}}{\Delta T} \tag{1}$$

whereas, in the solid,

$$y = \frac{y^*}{h} \quad \theta_w = \frac{T_w - T_{2\infty}}{\Delta T}. \tag{2}$$

x^* is the horizontal distance from one of the edges of the strip, y^* is the vertical distance measured from the middle of the strip toward the hot medium, and ψ_i^* are the stream functions defined in the usual way. Using the Boussinesq approximation, Darcy's law and the energy conservation equations are

$$\frac{\partial^2\psi_i}{\partial y_i^2} = \mp\frac{\partial\theta_i}{\partial x} \tag{3}$$

and

$$\frac{\partial\psi_i}{\partial y_i}\frac{\partial\theta_i}{\partial x} - \frac{\partial\psi_i}{\partial x}\frac{\partial\theta_i}{\partial y_i} = \frac{\partial^2\theta_i}{\partial y_i^2} \tag{4}$$

in the boundary layers. Equation (3) is obtained by taking the y-derivative of the horizontal component of Darcy's law, $u_i \equiv \partial\psi_i/\partial y_i = -\partial p_i/\partial x$, with the reduced pressure satisfying the hydrostatic balance $\partial p_i/\partial y_i = \pm\theta_i$ in the vertical direction. The upper signs in these equations correspond to having the hot medium below the wall, in which case the motion is from

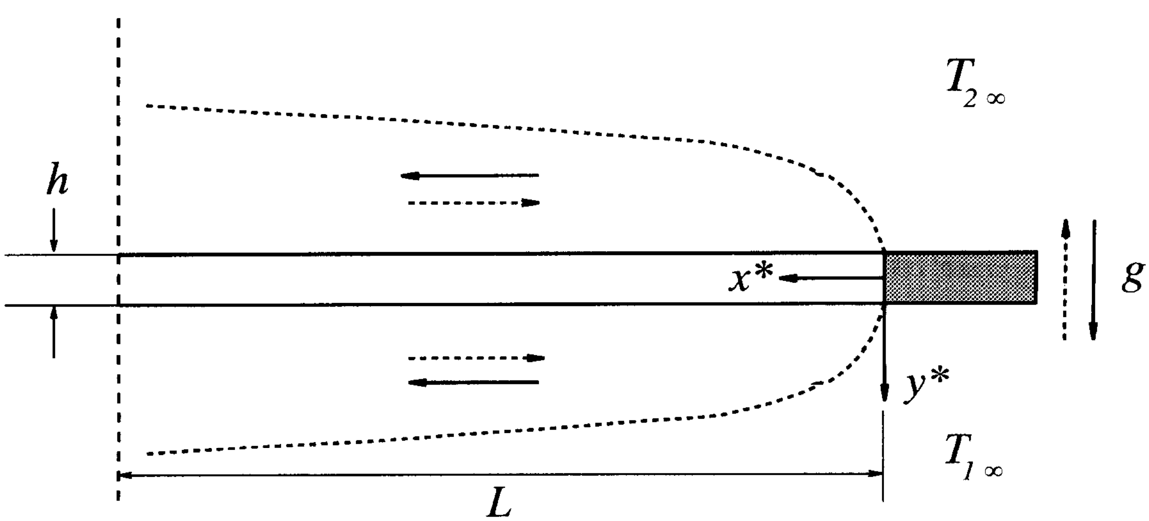

Fig. 1. Definition sketch. Solid arrows correspond to a hot medium below the wall and dashed arrows to a hot medium above the wall. Only half of the symmetric flow is depicted.

the edges toward the centre of the strip, as mentioned before. The lower signs correspond to having the hot medium above the wall, with the motion from the centre toward the edges (dashed arrows in Fig. 1). The non-dimensional heat conduction equation in the solid strip is

$$\varepsilon^2 \frac{\partial^2 \theta_w}{\partial x^2} + \frac{\partial^2 \theta_w}{\partial y^2} = 0 \tag{5}$$

and the boundary conditions for equations (3)–(5) are

$$\left.\begin{aligned} \psi_1 &= 0 \\ \theta_1(x,0) &= 1-\theta_w(x,\tfrac{1}{2}) \\ \partial\theta_1/\partial y_1 &= -\alpha\partial\theta_w/\partial y \end{aligned}\right\} \quad \text{at } y_1 = 0 \quad (y = \tfrac{1}{2}) \tag{6)–(8}$$

$$\left.\begin{aligned} \psi_2 &= 0 \\ \theta_2(x,0) &= \theta_w(x,-\tfrac{1}{2}) \\ \partial\theta_2/\partial y_2 &= -\alpha\beta\partial\theta_w/\partial y \end{aligned}\right\} \quad \text{at } y_2 = 0 \quad (y = -\tfrac{1}{2}) \tag{9)–(11}$$

$$\frac{\partial \psi_i}{\partial y_i} = \theta_i = 0 \quad \text{for } y_i \to \infty \tag{12}$$

$$\frac{\partial \theta_w}{\partial y_i} = 0 \quad \text{at } x = 0, 2 \tag{13}$$

plus appropriate conditions on the singularities of the boundary layer solutions at the edges. Thus, for $x \ll 1$, $u_i = O(x^{-1/3})$ and $y_c = O(x^{2/3})$ when the hot medium is below the wall [5, 6], and $u_i = O(x^{-1/2})$ and $y_c = O(x^{1/2})$ when it is above [7, 8], where y_c is the non-dimensional boundary layer thickness. The non-dimensional parameters appearing in the previous equations are

$$\alpha = \frac{k_w}{k_1}\frac{L}{h}\frac{1}{Ra_1^{1/3}} \quad \beta = \frac{k_1}{k_2}\left(\frac{Ra_1}{Ra_2}\right)^{1/3} \quad \varepsilon = \frac{h}{L} \tag{14}$$

where k_i and k_w are the thermal conductivities of the porous media and the solid wall. Here α is the ratio of the thermal resistance of the boundary layer in the hot medium to the thermal resistance of the wall, β is the ratio of the thermal resistances of the boundary layers in the cold and hot media, and ε is the aspect ratio of the strip, which is typically small.

The heat transport across the strip is measured by the overall Nusselt number:

$$\overline{Nu} = \frac{\bar{q}_w L}{k_1 \Delta T} = -Ra_1^{1/3}\int_0^1 \left(\frac{\partial \theta_1}{\partial y_1}\right)_0 \mathrm{d}x \tag{15}$$

where

$$\bar{q}_w = \frac{k_1}{L}\int_0^L \left(\frac{\partial T_1}{\partial y^*}\right)_{y^* = h/2} \mathrm{d}x^*$$

$$= \frac{k_2}{L}\int_0^L \left(\frac{\partial T_2}{\partial y^*}\right)_{y^* = -h/2} \mathrm{d}x^*.$$

This completes the formulation of the problem. In the remainder of this section, the asymptotic forms of the solution for large and small values of α will be described.

At the limit $\alpha \to \infty$, corresponding to very small thermal resistance of the wall compared to that of the boundary layers, equations (8) and (11) imply that the nondimensional temperature difference between the two faces of the wall is of order $1/\alpha \ll 1$. Neglecting such small differences in (7) and (10), the problem has a solution with uniform wall temperature ($\theta_w = \theta_0$ say) at leading order, of the form

$$\begin{aligned} \psi_1 &= (1-\theta_0)^{1/3}\Psi(x,(1-\theta_0)^{1/3}y_1) \\ \theta_1 &= (1-\theta_0)\Theta(x,(1-\theta_0)^{1/3}y_1) \end{aligned} \tag{16}$$

and

$$\psi_2 = \theta_0^{1/3}\Psi(x,\theta_0^{1/3}y_2) \quad \theta_2 = \theta_0\Theta(x,\theta_0^{1/3}y_2) \tag{17}$$

where $\Psi(x, Y)$ and $\Theta(x, Y)$ satisfy equations (3), (4) and (12) with the boundary conditions $\Psi = \Theta - 1 = 0$ at $Y = 0$. The solutions of these problems have been obtained elsewhere. When the hot medium is below the plate the solution is the adaptation of the self-similar solution of Stewartson to porous media [5]: $\Psi = x^{1/3}F(\eta)$, $\Theta = G(\eta)$, with $\eta = Y/x^{1/3}$ and

$$F'' - \tfrac{2}{3}\eta G' = 0 \quad G'' + \tfrac{1}{3}GG' = 0 \tag{18}$$

$$F(0) = G(0) - 1 = 0 \quad F'(\infty) = G(\infty) = 0$$

(primes denoting derivatives with respect to the self-similar variable η), for which $n_0 \equiv -\int_0^1(\partial\Theta/\partial Y)_{Y=0}\,\mathrm{d}x = -3G'(0) \approx 1.2897$. When the hot medium is above the plate the problem has a self-similar solution of a different type [8], for which $n_0 \equiv -\int_0^1(\partial\Theta/\partial Y)_{Y=0}\,\mathrm{d}x \approx 1.024$. Finally, θ_0 is obtained by equating the overall non-dimensional heat transfer from medium 1 to the plate and from the plate to medium 2, equal to $(1-\theta_0)^{4/3}n_0$ and $\theta_0^{4/3}n_0/\beta$, respectively [see equations (16) and (17)]. This condition yields

$$\theta_0 = \frac{\beta^{3/4}}{1+\beta^{3/4}} \tag{19}$$

and the overall Nusselt number is

$$\overline{Nu} = (1-\theta_0)^{4/3} n_0 \, Ra_1^{1/3} \tag{20}$$

At the opposite limit, $\alpha \to 0$, the thermal resistance of the wall is much larger than that of the boundary layers, and most of the temperature drop occurs across the wall, leading to $\theta_w = y + \frac{1}{2}$ in first approximation. The flow in the boundary layers can be found by solving equations (3), (4), (6), (8), (9), (11) and (12) with $\partial\theta_w/\partial y = 1$ in the right-hand sides of equations (8) and (11). The solutions are

$$\psi_1 = \alpha^{1/4}\tilde{\Psi}(x,\alpha^{1/4}y_1) \quad \theta_1 = \alpha^{3/4}\tilde{\Theta}(x,\alpha^{1/4}y_1) \tag{21}$$

and

$$\begin{aligned} \psi_2 &= (\alpha\beta)^{1/4}\tilde{\Psi}(x,(\alpha\beta)^{1/4}y_2) \\ \theta_2 &= (\alpha\beta)^{3/4}\tilde{\Theta}(x,(\alpha\beta)^{1/4}y_2) \end{aligned} \tag{22}$$

where $\tilde{\Psi}(x, Y)$ and $\tilde{\Theta}(x, Y)$ satisfy equations (3), (4) and (12) with the boundary conditions $\tilde{\Psi} = \partial\tilde{\Theta}/\partial Y + 1 = 0$ at $Y = 0$. Also in this case, a self-similar solution, of the form $\tilde{\Psi} = x^{1/2}\tilde{F}(\tilde{\eta})$, $\tilde{\Theta} = x^{1/2}\tilde{G}(\tilde{\eta})$, $\tilde{\eta} = Y/x^{1/2}$, exists when the hot medium is below the wall [6], giving $\tilde{\Theta}(x, 0) \approx 1.1643x^{1/2}$ and $m_0 \equiv \int_0^1 \tilde{\Theta}(x, 0)\,dx \approx 0.7762$, whereas when the hot medium is above the wall the numerical solution of the problem [8], which is not self-similar, gives $m_0 \equiv \int_0^1 \tilde{\Theta}(x, 0)\,dx \approx 1.2024$. Knowing the small values of θ_1 and θ_2 at the faces of the wall, equation (5) with conditions (7), (10) and (13) determines $\theta_w(x, y)$ to a better approximation. For $\varepsilon \ll 1$, longitudinal conduction can be neglected in equation (5) and the solution is

$$\theta_w = y + \tfrac{1}{2} - \alpha^{3/4}\tilde{\Theta}(x, 0)[y + \tfrac{1}{2} + \beta^{3/4}(y - \tfrac{1}{2})] \quad (23)$$

leading to an overall Nusselt number

$$\overline{Nu} = \alpha Ra_1^{1/3}\{1 - m_0\alpha^{3/4}(1 + \beta^{3/4})\}. \quad (24)$$

3. NUMERICAL RESULTS AND DISCUSSION

For $\alpha = O(1)$, problem (3)–(13) was solved numerically, taking $\varepsilon = 0$ in equation (5). The overall Nusselt numbers [equation (15)] computed from these numerical solutions for the cases of hot medium below and above the wall are given in Figs. 2 and 3, respectively, as functions of α for $\beta = 0.5$ and 1. Also plotted in Figs. 2 and 3 are the asymptotic results (20) and (24).

The heat transfer is somewhat higher when the hot medium is below the wall than when it is above. This can be traced to the stronger divergence of the heat flux at the edges: $\partial\theta_i/\partial y_i = O(x^{-2/3})$ for $x \ll 1$ in the first case, compared to $\partial\theta_i/\partial y_i = O(x^{-1/2})$ in the second case.

In both cases the Nusselt number increases with α. This is because a larger fraction of the total temperature drop from $T_{1\infty}$ to $T_{2\infty}$ occurs in the boundary layers as the thermal resistance of the solid decreases, leading to larger heat fluxes. The maximum Nusselt number is attained for $\alpha \to \infty$, being given by equation (20). In this limit the temperature of the solid also becomes uniform along the strip, owing to the fact that the two boundary layers grow at the same pace and from the same section of the strip, and thus longitudinal conduction in the solid plays no role in the present horizontal strip configuration (compare with the analysis of ref. 3 for a vertical wall). The temperature of the solid would depend on x for $\alpha \to \infty$ if the wall were inclined at an angle of order $Ra^{-1/3}$, and the longitudinal conduction in the solid would then become important for $\alpha = O(\varepsilon^{-2}) \gg 1$, making the temperature uniform and reducing the Nusselt number. Finally, for inclinations of the wall to the horizontal much larger than $Ra^{-1/3}$, the flow in the boundary layers would no longer be driven by the pressure gradient but by the component of the buoyancy parallel to the plate, and the results of ref. [3] become applicable by replacing g by its component parallel to the wall.

The Nusselt number defined in equation (15) increases as β decreases. For $\beta \to 0$ the thermal resist-

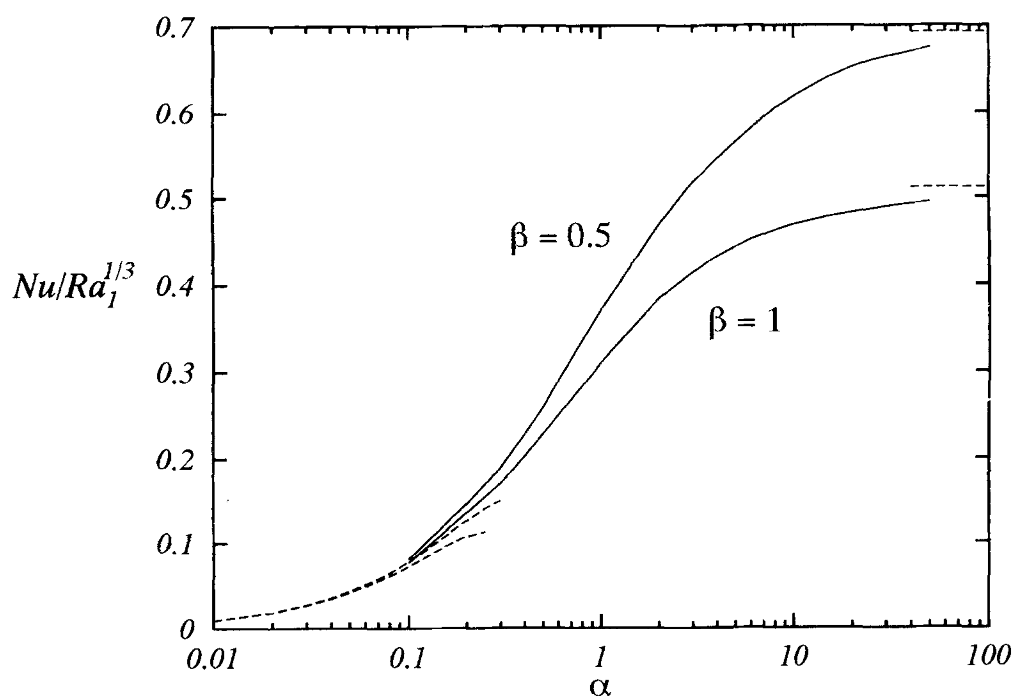

Fig. 2. Overall Nusselt number as a function of α for $\beta = 0.5$ and 1 for the case of a hot medium below the wall: results from the numerical integration of equations (3)–(13) (solid) and the asymptotic expressions (20) and (24) (dashed).

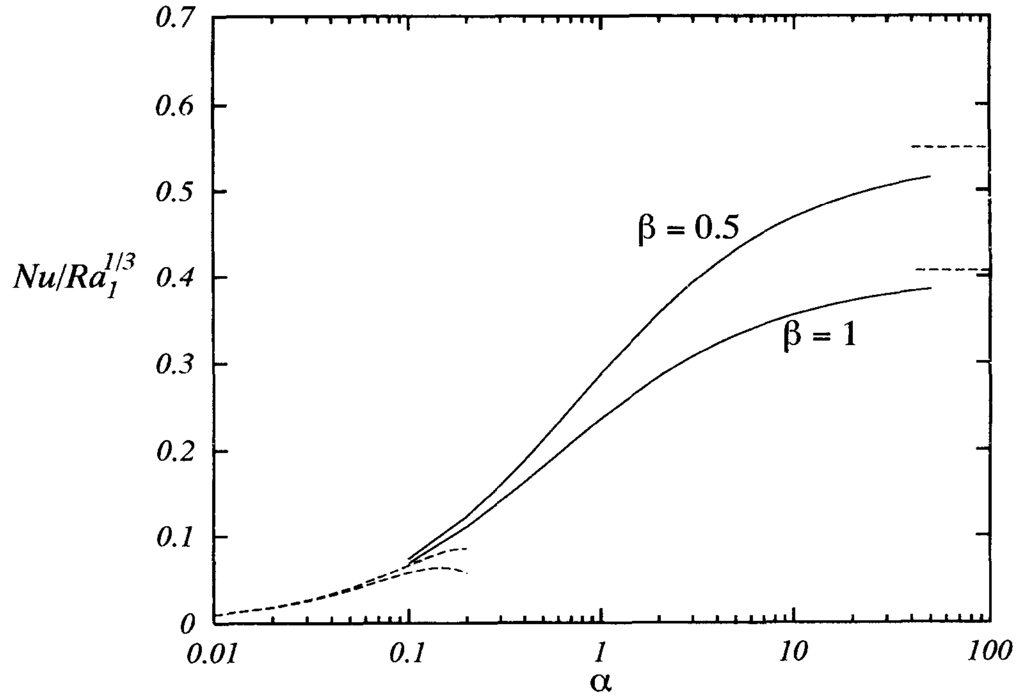

Fig. 3. Overall Nusselt number as a function of α for $\beta = 0.5$ and 1 for the case of a hot medium above the wall: results from the numerical integration of equations (3)–(13) (solid) and the asymptotic expressions (20) and (24) (dashed).

ance of the boundary layer in the cold medium becomes negligible, and the non-dimensional temperature of the solid surface facing that medium tends to zero. Analogously, the thermal resistance of the boundary layer in the hot medium becomes negligible, and the non-dimensional temperature of the solid surface facing that medium tends to 1 for $\beta \to \infty$. It is worth noticing that the results for values of β smaller and larger than the unity can be easily related to each other using the invariance properties of problem (3)–(13). Thus, as can be easily verified, if $\overline{Nu}/Ra_1^{1/3} = f(\alpha)$ for a certain $\beta = \beta_1$, the Nusselt number for $\beta = 1/\beta_1$ is $\overline{Nu}/Ra_1^{1/3} = \beta_1 f(\alpha/\beta_1)$.

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